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Successive Approximations to Solutions of Volterra Integral Equations

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1. INTRODUCTION

Several authors have considered the problem of best approximate solutions of nonlinear differential equations [1, 2, 4, 5]. Recently, many of these results have been extended to integral or integro-differential equations [6, 8]. In the case of differential equations, a successive approximation method has been developed to find best approximate solutions [3, 7]. This method, being based on the Remes algorithm, is too inefficient for use with integral equations. The objectives of this paper are to modify the algorithm of [3, 7] for use on integral equations, to use the successive approximations to prove an existence theorem for solutions of Volterra integral equations, and to provide error estimates for the successive approximations.

2. PRELIMINARY DEFINITIONS

Consider the Volterra integral equation

$$x(t) = F(t) + \int_0^t G(t, s, x(s))\, ds \tag{2.1}$$

for $t \in I = [0, \sigma]$, where F and G are continuous on I and $I^2 \times R$, respectively.

For $k = 0, 1, 2,\ldots$, define the functions

$$\phi_{k+1}(t) = \frac{2}{(\pi\sigma)^{1/2}}\, T_k\left(\frac{2t-\sigma}{\sigma}\right), \qquad k \neq 0,$$

$$= \left(\frac{2}{\pi\sigma}\right)^{1/2} T_0\left(\frac{2t-\sigma}{\sigma}\right), \quad k = 0,$$

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where T_k is the kth degree Chebyshev polynomial. From the relation

$$\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{(1-x^2)^{1/2}}\, dx = \begin{cases} 0, & m \neq n, \\ \pi/2, & n = m \neq 0, \\ \pi, & n = m = 0, \end{cases}$$

we see that the set $\{\phi_k\}_{k=1}^{\infty}$ is orthonormal on I with respect the inner product

$$\langle f, g\rangle = \int_0^{\sigma} \frac{f(t)\, g(t)}{(1 - ((2t - \sigma)/\sigma)^2)^{1/2}}\, dt.$$

Our approximating set will be

$$G_k = \operatorname{span}\{\phi_1, \phi_2, ..., \phi_k\}.$$

For $x \in C(I)$, we define the operator

$$L[x](t) = F(t) + \int_0^t G(t, s, x(s))\, ds$$

and the norms

$$\| x \| = \max_I | x(t)|$$

and

$$\| x \|_2 = \langle x, x\rangle^{1/2}.$$

We note that

$$\| x \|_2 \leqslant \left(\frac{\sigma\pi}{2}\right)^{1/2} \| x \|. \tag{2.2}$$

3. Successive Approximations

As a first approximate solution of (2.1) from G_k, we choose $p_{1,k}(t) = c_1\phi_1(t)$ so that $p_{1,k}(0) = F(0)$. Thus we set $c_1 = (\pi\sigma/2)^{\frac{1}{2}} F(0)$. Then for $n \geqslant 1$ and $p_{n,k} \in G_k$ given, we select $p_{n+1,k} \in G_k$ to solve the minimization problem

$$\inf_{p \in G_k} \| p - L[p_{n,k}]\|_2 .$$

Since $L[p_{n,k}]$ is a known continuous function defined on I, we set $p_{n+1,k} = \sum_{i=1}^{k} \langle \phi_i, L[p_{n,k}]\rangle\, \phi_i$. This process defines a sequence $\{p_{n,k}\}_{n=1}^{\infty} \subseteq G_k$.

4. Convergence of the Sequence $\{p_{n,k}\}_{n=1}^{\infty}$

Set $K = \| F\| + 2$ and define $B_k = \{p \in G_k : \| p \| \leqslant K\}$. Since G is continuous, there is a constant M satisfying $| G(t, s, x)| \leqslant M$ whenever $(t, s, x) \in I^2 \times [-K, K]$.

We now impose a condition on F and G to ensure that the sequence $\{p_{n,k}\}_{n=1}^{\infty}$ will have a convergent subsequence for σ (independent of k) not too large.

Condition 1. Suppose F satisfies the Dini–Lipschitz condition

$$\lim_{\delta \to 0} \omega(F; [0, \sigma]; \delta) \log \delta = 0,$$

where ω is the modulus of continuity of F (see [9, p. 14]). Further suppose that

$$\omega(G(\cdot, x_1, x_2);[0, \sigma]; \delta) \log \delta$$

converges uniformly on $I \times [-K, K]$ to zero as δ tends to zero.

Define the error function

$$e_{n,k} = p_{n+1,k} - L[p_{n,k}].$$

LEMMA 1. *The function $e_{n,k}$ satisfies*

$$\| e_{n,k} \| \leqslant \left(4 + \frac{4}{\pi^2} \log(k-1)\right) 6\omega \left(L[p_{n,k}]; [0, \sigma]; \frac{\sigma}{2(k-1)}\right).$$

Proof. Transforming the independent variable, we have

$$\| e_{n,k} \| = \max_{[-1,1]} \left| p_{n+1,k} \left(\frac{\sigma(t+1)}{2}\right) - L[p_{n,k}] \left(\frac{\sigma(t+1)}{2}\right)\right|.$$

Let $H(t) = L[p_{n,k}](\sigma(t+1)/2)$ and $p(t) = p_{n+1,k}(\sigma(t+1)/2)$. Then there are constants a_i ($i = 0, 1,..., k-1$) so that $p(t) = \sum_{i=0}^{k-1} a_i T_i(t)$. From the way that $p_{n+1,k}$ was chosen, it is clear that p is the best least-squares approximation to H on the interval $[-1, 1]$ by Chebyshev polynomials. If we let q represent the best uniform approximation to H on the interval $[-1, 1]$ by polynomials of degree at most $k-1$, we have (see [9, p. 61])

$$\begin{aligned} \| e_{n,k} \| &= \max_{[-1,1]} | p(t) - H(t)| \\ &\leqslant \left[4 + \frac{4}{\pi^2} \log(k-1)\right] \max_{[-1,1]} | q(t) - H(t)| \\ &= \left[4 + \frac{4}{\pi^2} \log(k-1)\right] \max_{[0,\sigma]} \left| q \left(\frac{2t-\sigma}{\sigma}\right) - L[p_{n,k}]\right|. \end{aligned}$$

Since $q((2t-\sigma)/\sigma)$ is the best uniform approximation to $L[p_{n,k}]$ on the interval $[0, \sigma]$ by polynomials of degree at most $k-1$, we use Jackson's theorem [9, p. 22] to conclude that

$$\| e_{n,k} \| \leqslant \left[4 + \frac{4}{\pi^2} \log(k-1)\right] 6\omega \left(L[p_{n,k}]; [0, \sigma]; \frac{\sigma}{2(k-1)}\right).$$

LEMMA 2. *There is a positive number $\bar{\sigma}$, independent of k, such that $\sigma \leqslant \bar{\sigma}$ implies that*

$$6\left[4 + \frac{4}{\pi^2}\log(k-1)\right]\omega\left(L[p];\ [0,\sigma];\ \frac{\sigma}{2(k-1)}\right) \leqslant 1$$

for all $p \in B_k$.

Proof. We first consider the modulus of continuity of $L[p]$. Let $t_1, t_2 \in I$ with $|t_1 - t_2| \leqslant \sigma/(2(k-1))$ and let $s^* \in I$ satisfy

$$\begin{aligned}&|G(t_1, s^*, p(s^*)) - G(t_2, s^*, p(s^*))|\\ &\quad= \max_{s\in I}|G(t_1, s, p(s)) - G(t_2, s, p(s))|.\end{aligned}$$

Then

$$\begin{aligned}&|L[p](t_1) - L[p](t_2)|\\ &\quad\leqslant |F(t_1) - F(t_2)| + \left|\int_0^{t_1} G(t_1, s, p(s))\,ds - \int_0^{t_2} G(t_2, s, p(s))\,ds\right|\\ &\quad= |F(t_1) - F(t_2)|\\ &\qquad+ \left|\int_0^{t_1}(G(t_1, s, p(s)) - G(t_2, s, p(s))\,ds + \int_{t_2}^{t_1} G(t_2, s, p(s))\,ds\right|\\ &\quad\leqslant |F(t_1) - F(t_2)| + t_1\,|G(t_1, s^*, p(s^*)) - G(t_2, s^*, p(s^*))|\\ &\qquad+ |t_1 - t_2|\,M.\end{aligned}$$

Taking suprema, we have

$$\begin{aligned}&\omega\left(L[p];\ [0,\sigma];\ \frac{\sigma}{2(k-1)}\right)\\ &\quad\leqslant \omega\left(F;\ [0,\sigma];\ \frac{\sigma}{2(k-1)}\right)\\ &\qquad+ \sigma\omega\left(G(\cdot, s^*, p(s^*));\ [0,\sigma];\ \frac{\sigma}{2(k-1)}\right) + \frac{\sigma}{2(k-1)}M\\ &\quad\leqslant \left(\frac{\sigma}{2}+1\right)\omega\left(F;\ [0,\sigma];\ \frac{1}{k-1}\right)\\ &\qquad+ \sigma\left(\frac{\sigma}{2}+1\right)\omega\left(G(\cdot, s^*, p(s^*));\ [0,\sigma];\ \frac{1}{k-1}\right) + \frac{\sigma}{2(k-1)}M.\end{aligned}$$

In the above inequality we employed the fact [9, p. 15] that for $\delta > 0$, $\omega(f;\ [a,b];\ \delta\epsilon) \leqslant (\delta+1)\,\omega(f;\ [a,b];\ \epsilon)$. From condition 1 and the fact that the left member of above inequality is monotone increasing in σ, we have

$$\lim_{k\to\infty}\omega\left(L[p];\ [0,\sigma'];\ \frac{\sigma'}{2(k-1)}\right)\log(k-1) = 0$$

uniformly for $\sigma' \in [0, \sigma]$. Also, using the continuity of F and the bound on G, we have

$$\lim_{\sigma \to 0} \omega \left(L[p]; [0, \sigma]; \frac{\sigma}{2(k-1)} \right) = 0$$

for $k \geqslant 2$ and for each $p \in B_k$. This limit is uniform in $p \in B_k$.

We now choose N so that $k \geqslant N$ and $\sigma' \in [0, \sigma]$ implies

$$\left[4 + \frac{4}{\pi^2} \log(k-1) \right] \omega \left(L[p]; [0, \sigma']; \frac{\sigma'}{2(k-1)} \right) \leqslant 1.$$

For $k = 2, 3, \ldots, N-1$, choose δ_k such that $\sigma \leqslant \delta_k$ and $p \in B_k$ implies

$$6 \left[4 + \frac{4}{\pi^2} \log(k-1) \right] \omega \left(L[p]; [0, \sigma]; \frac{\sigma}{2(k-1)} \right) \leqslant 1. \tag{4.1}$$

Then if $\bar{\sigma} = \min\{\delta_1, \delta_2, \ldots, \delta_{N-1}\}$ and $\sigma \leqslant \bar{\sigma}$, we have inequality (4.1) holding for all $k \geqslant 2$ and for all $p \in B_k$.

We now impose the final condition to insure that the sequence of successive approximations possesses a cluster point.

Condition 2. Let σ satisfy $\sigma M \leqslant 1$ and suppose σ is sufficiently small that (4.1) holds for each $p \in B_k$, $k \geqslant 2$.

THEOREM 1. *If conditions* 1 *and* 2 *hold, then the sequence* $\{p_{n,k}\}_{n=1}^{\infty}$ *has a convergent subsequence for* $k \geqslant 2$.

Proof. We will use iduction to show that $\{p_{n,k}\}_{n=1}^{\infty} \subseteq B_k$, a uniformly bounded subset of a k-dimensional linear space. Clearly $p_{1,k} = F(0) \in B_k$. Suppose $p_{n,k} \in B_k$. Since $p_{n+1,k} = L[p_{n,k}] + e_k$, we can use Lemmas 1 and 2 and conditions 1 and 2 to conclude that

$$\begin{aligned} \| p_{n+1,k} \| &\leqslant \| L[p_{n,k}] \| + \| e_k \| \\ &\leqslant \| F \| + \sigma M + 1 \\ &\leqslant \| F \| + 2 \end{aligned}$$

and consequently $p_{n+1,k} \in B_k$. Since B_k is compact, the sequence $\{p_{n,k}\}_{n=1}^{\infty}$ must have a convergent subsequence.

THEOREM 2. *Suppose G satisfies the following local Lipschitz condition. For each* $B > 0$, *there exists* $\lambda_B > 0$ such that

$$| G(t, s, y_1) - G(t, s, y_2) | \leqslant \lambda_B \, | y_1 - y_2 |,$$

whenever $t \in I$, $s \in I$, $| y_1 | \leqslant B$, *and* $| y_2 | \leqslant B$. *Let λ be the Lipschitz constant corresponding to* $K = \| F \| + 2$. *Then if conditions* 1 *and* 2 *hold and if* $2^{1/2} \lambda k \bar{\sigma} < 1$, *the sequence* $\{p_{n,k}\}_{n=1}^{\infty}$ *converges.*

Proof. Define the mapping T as follows. Given $p \in G_k$, let $T(p)$ solve the approximation problem

$$\inf_{q \in G_k} \| q - L[p] \|_2 .$$

Thus

$$T(p) = \sum_{i=1}^{k} \langle \phi_i , L[p] \rangle \phi_i .$$

We now show that T is a contraction mapping. Let $p, q \in B_k$. Using the Cauchy–Schwarz inequality and inequality (2.2), we have

$$\begin{aligned} \| T(p) - T(q) \| &= \left\| \sum_{i=1}^{k} \langle \phi_i , L[p] - L[q] \rangle \phi_i \right\| \\ &\leqslant (2k/(\pi\sigma)^{1/2}) \| L[p] - L[q] \|_2 \\ &\leqslant 2^{1/2} k \| L[p] - L[q] \| \\ &\leqslant 2^{1/2} \lambda k \sigma \| p - q \|. \end{aligned}$$

Since $2^{1/2}\lambda k \sigma < 1$, the mapping T is a contraction and since G_k is a complete metric space, T has a unique fixed point. Furthermore this fixed point is the limit of the sequence $\{p_{n,k}\}_{n=1}^{\infty}$.

THEOREM 3. *If the sequence* $\{p_{n,k}\}_{n=1}^{\infty}$ *converges to* p_k, *then* p_k *solves the minimization problem* $\inf_{p \in G_k} \| p - L[p_k] \|_2$.

Proof. Letting $n \to \infty$ in the equation

$$p_{n+1,k} = \sum_{i=1}^{k} \langle \phi_i , L[p_{n,k}] \rangle \phi_i$$

yields

$$p_k = \sum_{i=1}^{k} \langle \phi_i , L[p_k] \rangle \phi_i .$$

It should be noted that a σ independent of k may be found satisfying the conditions of Theorem 1. This ensures that the sequences $\{p_{n,k}\}_{n=1}^{\infty}$ $(k = 2, 3, \ldots)$ each have a cluster point for a fixed σ.

Throughout the remainder of this paper, we assume the following.

Condition 3. The sequences $\{p_{n,k}\}_{n=1}^{\infty}$ $(k = 2, 3, \ldots)$ actually converge to p_k $(k = 2, 3, \ldots)$, respectively, for some fixed σ.

5. Existence of Solutions of (2.1)

Define the error function

$$e_k = p_k - L[p_k].$$

THEOREM 4. *If conditions* 1 *and* 3 *hold, then* $\lim_{k\to\infty} \| e_k \| = 0$.

Proof. From Theorem 3 and the proof of Lemma 1, we see that

$$\| e_k \| \leqslant (4 + (4/\pi^2) \log(k - 1))\, 6\omega(L[p_k]; [0, \sigma]; \sigma/(2(k - 1))).$$

The result now follows from condition 1 and a slight modification of the proof of Lemma 2.

THEOREM 5. *If conditions* 1 *and* 3 *hold, then the sequence* $\{p_k\}_{k=1}^{\infty}$ *has a subsequence which converges uniformly on* I *to a function* y. *Moreover, the function* y *is a solution of* (2.1).

Proof. Define the functions

$$v_k(t) = \int_0^t G(t, s, p_k(s))\, ds \qquad \text{for} \quad t \in I.$$

Since G is uniformly continous on $I^2 \times [-K, K]$, the family $\{v_k\}_{k=1}^{\infty}$ is uniformly bounded and equicontinuous. By Ascoli's theorem, there is a subsequence $\{v_{k(l)}\}_{l=1}^{\infty}$ which converges uniformly on I to some function v. Set $y = v + F$ and consider the equation

$$p_{k(l)}(t) = F(t) + \int_0^t G(t, s, p_{k(l)}(s))\, ds + e_{k(l)}(t). \tag{5.1}$$

Applying the triangle inequality, we find that

$$\| p_{k(l)} - y \| \leqslant \| v - v_{k(l)} \| + \| e_{k(l)} \|.$$

Now letting $l \to \infty$ and using Theorem 4, we have

$$\lim_{l\to\infty} \| p_{k(l)} - y \| = 0.$$

Finally, letting $l \to \infty$ in (5.1) we see that y is a solution of (2.1).

6. Error Estimates and Convergence

Let y be the solution of (2.1) given by Theorem 5. Combining the equations

$$y(t) = F(t) + \int_0^t G(t, s, y(s))\, ds$$

and

$$p_k(t) = F(t) + \int_0^t G(t, s, p_k(s))\, ds + e_k(t),$$

we have

$$y(t) - p_k(t) = \int_0^t (G(t, s, y(s)) - G(t, s, p(s)))\, ds - e_k(t).$$

Since each $p_k \in B_k$, we have $\| p_k \| \leqslant K$ for each k. Thus $\| y \| \leqslant K$ and

$$| y(t) - p_k(t)| \leqslant \int_0^t \lambda \mid y(s) - p_k(s)| \, ds + \| e_k \|$$

once again assuming the local Lipschitz condition stated in Theorem 2.

Gronwall's inequality implies

$$\| y - p_k \| \leqslant \| e_k \| \exp(\lambda\sigma).$$

Combining the above inequality with Theorem 4, we now see that the sequence $\{p_k\}_{k=1}^{\infty}$ actually converges to y. We summarize this in the following theorem.

THEOREM 6. *If conditions* 1 *and* 3 *and the local Lipschitz condition stated in Theorem* 4 *hold, then the sequence* $\{p_k\}_{k=1}^{\infty}$ *converges uniformly on* I *to a solution* y *of* (2.1).

7. Numerical Examples

In each of the following examples, the approximate solutions from G_k are expressed as polynomials.

EXAMPLE 1. $x(t) = 1 + \int_0^t x(s)\, ds$, $t \in [0, 1]$. After 12 iterations,

$$p_4(t) = 0.9994 + 1.0169t + 0.4208t^2 + 0.2805t^3.$$

The solution is $y(t) = \exp(t)$ and $\max_{[0,1]} | p_4(t) - \exp(t)| = 0.0005$. In this elementary example, the Picard iterates can easily be computed. The fourth Picard iterate is $q_4(t) = 1 + t + 0.5t^2 + 0.1667t^3$ and $\max_{[0,1]} |q_4(t) - \exp(t)| = 0.05$.

EXAMPLE 2. $x(t) = 1 + t + \int_0^t (t - s)\, e^{-s} x^2(s)\, ds$, $t \in [0, 1]$. After seven iterations,

$$p_4(t) = 0.9993 + 1.0174t + 0.4204t^2 + 0.2806t^3.$$

Again, the solution is exp(t). The maximum error is

$$\max_{[0,1]} |p_4(t) - \exp(t)| = 0.00066.$$

The integral equation in this example may be written as the second-order differential equation

$$x''(t) = e^{-t}x^2(t), \qquad t \in [0, 1],$$

with initial conditions $x(0) = 1$ and $x'(0) = 1$. Applying the SAS algorithm [3], we find that the SAS of degree 3 or less is

$$p_4^*(t) = 1 + t + 0.4422t^2 + 0.2916t^3.$$

The maximum error is

$$\max_{[0,1]} |p_4^*(t) - \exp(t)| = 0.016.$$

EXAMPLE 3. $x(t) = F(t) + \int_0^t x^2(s)\,ds,\ t \in [0, 1]$, where

$$\begin{aligned} F(t) &= t, && 0 \leqslant t \leqslant 0.25, \\ &= 0.25, && 0.25 < t \leqslant 1. \end{aligned}$$

The solution of this equation is

$$\begin{aligned} y(t) &= \tan t, && 0 \leqslant t \leqslant 0.25, \\ &= 1/(4.1663 - t), && 0.25 < t \leqslant 1. \end{aligned}$$

After seven iterations, we have

$$p_4(t) = -0.0059 + 1.3618t - 2.0782t^2 + 1.0456t^3.$$

The maximum error is

$$\max_{[0,1]} |p_4(t) - y(t)| = 0.03.$$

The larger error is expected due to the lack of smoothness in F. The maximum error of 0.03 is assumed at approximately $t = 0.25$. Throughout most of the interval, the error is much smaller, e.g., $\sup_{[0,1]-[0.1,0.3]} |p_4(t) - y(t)| = 0.015$.

8. CONCLUSIONS

The method of successive approximation, based on best least-squares approximation, provides computable continous approximate solutions for the Volterra integral equation. In addition, the successive approximations have been used to verify an existence theorem for solutions of Volterra integral equations. Although condition 3 appears difficult to check, no example has been found which violates this condition. In fact, it is conjectured that if $\{p_{n,k}\}_{n=1}^{\infty}$ converges for some σ, then $\{p_{n,k+1}\}_{n=1}^{\infty}$ also converges for the same σ. The SAS algorithm [3] depends on a condition similar to condition 3.

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